Name (IN CAPITALS): **Version #1**

Instructor: Dora The Explorer

Math 10560 Exam 3
Apr. 22 2025

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off (and Put Away) all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 15 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)
11.	(•)	(b)	(c)	(d)	(e)
12.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
13.	
14.	
15.	
16.	
Total	

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1.	(a)	(b)	(c)	(d)	(e)
2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice ________________

13. ________________

14. ________________

15. ________________

16. ________________

Total ________________

Multiple Choice

1.(7pts) Use the comparison test or limit comparison test to determine which of the following series are convergent:

$$(I)\ \sum_{n=2}^{\infty} \frac{e^{1/n}+1}{\sqrt{n}} \qquad (II)\ \sum_{n=2}^{\infty} \frac{n^2+2n+1}{n^3+2n^2+1} \qquad (III)\ \sum_{n=1}^{\infty} \frac{1}{n3^n}$$

Note that $e^{1/n} \geqslant 1$, so $(e^{1/n}+1)/\sqrt{n} \geqslant 2/\sqrt{n} = 2/n^{1/2}$, and $\sum_n 1/n^{1/2}$ diverges by p-test, so so (I) diverges by the comparison test. For (II), taking the leading coefficients of the numerator and denominator gives $n^2/n^3 = 1/n$, and $\sum_n 1/n$ diverges by p-test, so (II) diverges by the limit comparison test. For (III), note that $3^n \geqslant n$ for all n, so $1/n3^n \leqslant 1/n^2$, and $\sum_n 1/n^2$ converges by p-test, so (III) converges by the comparison test.

(a) Only III converges

(b) Only I and III converge

(c) Only I and II converge

(d) All three converge

(e) All three diverge

2.(7pts) Test the following series for absolute convergence, conditional convergence or divergence:

$$\text{(I)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \text{(II)} \sum_{n=0}^{\infty} (-1)^n \frac{3}{2^n}$$

(I) does not converge absolutely, since $\sum_n 1/\sqrt{n} = \sum_n 1/n^{1/2}$ diverges by p-test. However $1/\sqrt{n}$ is a decreasing sequence that tends to 0, so the series in (I) converges by the alternating series test. Hence (I) conditionally converges. For (II), note that $\lim_{n\to\infty} n^2/2^n = 0$, and $\sum_n 1/n^2$ converges by p-test, so $\sum_n 1/2^n$ converges by the limit comparison test and hence (II) converges absolutely.

(a) (I) converges conditionally and (II) converges absolutely.

(b) Both converge conditionally.

(c) Both converge absolutely.

(d) (I) converges absolutely and (II) diverges.

(e) (I) converges conditionally and (II) diverges.

3.(7pts) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{16+n^2}.$$

Letting $f(x) = \dfrac{x}{16+x^2}$, which of the following statements is true?

Note that $\lim_{n\to\infty} n/(16+n^2) = 0$, for example by applying L'Hopital, and that the derivative of $f(x)$ is $(16-x^2)/(16+x^2)^2$ by the quotient rule, which is negative for all $x > 4$, so that $f(x)$ is decreasing for $x > 4$. So the series converges by the alternating series test.

(a) The series converges by the Alternating Series Test.

(b) The series converges by the Limit Comparison Test (compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

(c) The series diverges by the Limit Comparison Test (compared with $\sum_{n=1}^{\infty} \frac{1}{n}$).

(d) The series converges because $\lim_{x\to\infty} f(x) = 0$.

(e) The series diverges by the Alternating Series Test because $f'(x) > 0$ for $x = 1$ and $x = 2$.

4.(7pts) Consider the following series

$$(I) \quad \sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 \cdot e^n} \qquad\qquad (II) \quad \sum_{n=1}^{\infty} \left(\frac{n+1}{2n^2+1}\right)^n.$$

Which of the following statements is true?

Note that $(n+1)!/n^2 = (n+1)n(n-1)!/n^2 \geqslant (n-1)!$. Also, for $k > 5$, $k! \geqslant e^k$. So $\lim_{n\to\infty}(n+1)!/(n^2 e^n) = \infty$, and hence (I) diverges by the divergence test. On the other hand, taking $a_n = ((n+1)/(2n^2+1))^n$, then $\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty}(n+1)/(2n^2+1) = 0$, so that (II) converges by the root test.

(a) (I) diverges and (II) converges.

(b) They both diverge.

(c) They both converge.

(d) (I) converges and (II) diverges.

(e) (I) converges conditionally and (II) diverges.

5.(7pts) Consider the following series

$$(I)\quad \sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+2}} \qquad (II)\quad \sum_{n=1}^{\infty} (-1)^n 2^{1/n} \qquad (III)\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Which of the following statements is true?

For (I), taking the leading terms of the numerator and denominator gives $2n/\sqrt{n^3} = 2/\sqrt{n}$, and $\sum_n 2/\sqrt{n}$ diverges by the p-test, so (I) diverges by the limit comparison test. For (II), note that $\lim_{n\to\infty} 2^{1/n} = 1 \neq 0$, so (II) diverges by the divergence test. (III) converges by the p-test.

(a) (I) diverges, (II) diverges, and (III) converges.

(b) (I) converges, (II) diverges, and (III) converges.

(c) They all converge.

(d) (I) converges, (II) converges, and (III) diverges.

(e) They all diverge.

6.(7pts) Find the radius of convergence R for the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{5^n n}.$$

Let $a_n = (x+1)^n/5^n n$. Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{n+1}}{5^{n+1}(n+1)} \frac{5^n n}{(x+1)^n}\right| = \left|\frac{x+1}{5}\right| \left|\frac{n}{n+1}\right| \to_{n\to\infty} \frac{|x+1|}{5},$$

so the radius of convergence is 5.

(a) $R = 5$

(b) $R = 1$

(c) $R = 0$

(d) $R = \infty$

(e) $R = 2$

7.(7pts) Find a power series representation for the function

$$\frac{x^2}{1+8x^3}$$

in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Recall that $\frac{1}{1-r} = \sum_n r^n$ for $|r| < 1$. So

$$\frac{x^2}{1+8x^3} = x^2 \cdot \frac{1}{1+8x^3} = x^2 \sum_n (-8x^3)^n = x^2 \sum_n (-1)^n 8^n x^{3n} = \sum_n (-1)^n 8^n x^{3n+2}.$$

(a) $\sum_{n=0}^{\infty} (-1)^n 8^n x^{3n+2}$

(b) $\sum_{n=0}^{\infty} 8^n x^{3n+2}$

(c) $\sum_{n=0}^{\infty} (-1)^n 8^n x^{3n}$

(d) $\sum_{n=0}^{\infty} 8x^{3n+2}$

(e) $\sum_{n=0}^{\infty} (-1)^n 8^{3n} x^{3n+2}$

8.(7pts) Consider the function $f(x)$ defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n n!}, \qquad -\infty < x < \infty.$$

Which of the following gives a power series expansion for $f'(x)$?

Note that

$$\frac{d}{dx} \frac{(-1)^n x^{2n}}{3^n n!} = \frac{(-1)^n}{3^n n!} \frac{d}{dx} x^{2n} = \frac{(-1)^n}{3^n n!} 2n x^{2n-1} = \frac{(-1)^n 2n x^{2n-1}}{3^n n!},$$

so integrating the series term-by-term gives (a).

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{3^n n!}$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n n x^{2n-1}}{3^n n!}$.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{3^n (2n+1)!}$.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{3^{2n-1} (2n-1)!}$.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{2n+1} (2n+1)!}$.

9.(7pts) Find the following limit using power series

$$\lim_{x\to 0} \frac{e^{(-x^{10})} - 1 + x^{10}}{x^{20}}$$

Note that $e^{-x^{10}} = \sum_n \frac{(-x^{10})^n}{n!} = 1 - x^{10} + \frac{x^{20}}{2} + \sum_{n\geqslant 3}(-1)^n \frac{x^{10n}}{n!}$, so that

$$e^{-x^{10}} - 1 + x^{10} = \frac{x^{20}}{2} + \sum_{n\geqslant 3}(-1)^n \frac{x^{10n}}{n!}.$$

for $n \geqslant 3$, we have $10n - 20 \geqslant 10 \geqslant 1$, so that $\lim_{x\to 0} \frac{1}{x^{20}} \frac{x^{10n}}{n!} = \lim_{x\to 0} \frac{x^{10n-20}}{n!} = 0$. So all of the terms for $n \geqslant 3$ vanish, and all that is left is $\lim_{x\to 0} \frac{1}{x^{20}} \frac{x^{20}}{2} = \frac{1}{2}$.

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1 (e) $-\frac{1}{6}$

10.(7pts) The following is the fourth order Taylor polynomial of a function $f(x)$ at $a = 3$.

$$2 + \sqrt{2}(x-3) + 10(x-3)^2 + 5(x-3)^3 + 3(x-3)^4$$

What is $f^{(3)}(3)$?

Recall that the degree 3 monomial of a Taylor polynomial centered at a is $\frac{f^{(3)}(a)}{3!}(x-a)^3 = \frac{f^{(3)}(a)}{6}(x-a)^3$. So we have $\frac{f^{(3)}(3)}{6} = 5$, so $f^{(3)}(3) = 6 \cdot 5 = 30$.

(a) 30 (b) 5 (c) 15 (d) 0 (e) 1

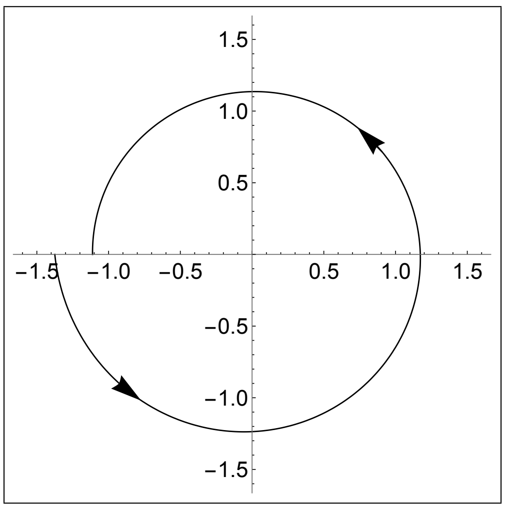
11.(7pts) Which of the following is a graph of the parametric curve defined by

$$x = e^{1/t}\cos(t) \qquad y = e^{1/t}\sin(t)$$

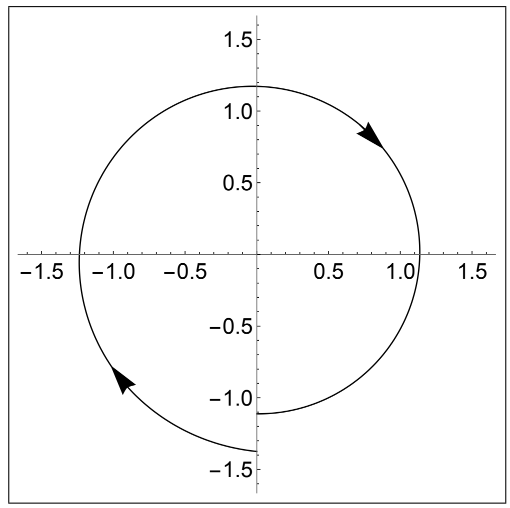
for $\pi \leq t \leq 3\pi$?

We compute a few points. At $t = \pi$, we have $(x, y) = (-e^{1/\pi}, 0)$. At $t = 2\pi$, we have $(x, y) = (e^{1/2\pi}, 0)$. At $x = 3\pi$, we have $(x, y) = (-e^{1/3\pi}, 0)$. We note that $e^{1/\pi} > e^{1/2\pi} > e^{1/3\pi}$. So this narrows down the possibilities to (a) and (c). But also at $t = \pi/2$, $(x, y) = (0, e^{2/\pi})$, and $e^{2/\pi} > 0$, so this rules out (c).

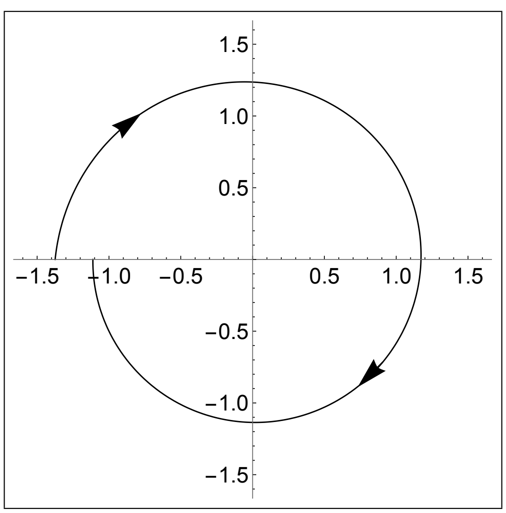
(a)



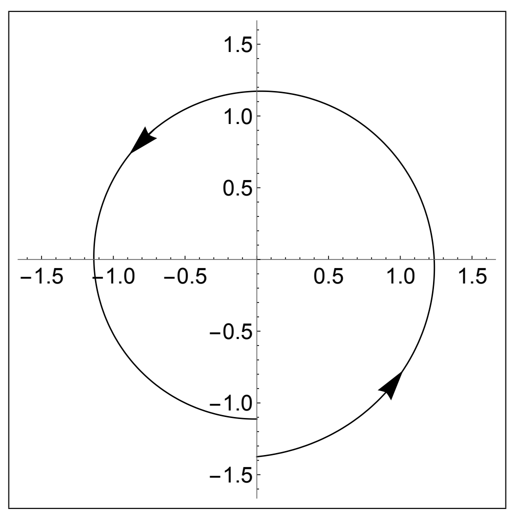
(b)



(c)

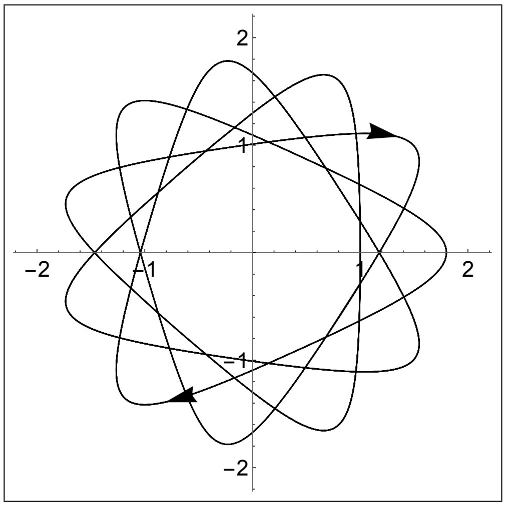


(d)



8. Initials: ____________

(e)



12.(7pts) Use a well known power series to find the sum of the following series:

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}$$

We recall that $\cos(x) = \sum_n (-1)^n \frac{x^{2n}}{(2n)!}$. So the sum in question is $\cos(\pi) = -1$.

(a) -1 (b) 1 (c) 0 (d) e^{π} (e) $e^{2\pi}$

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

13.(12pts) Find the radius of convergence and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n(n+1)}.$$

If, in the course of the solution, you test for convergence of a series, please state clearly which test you are using.

Let $a_n = (x-2)^n/3^n(n+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(x-2)^{n+1}}{3^{n+1}(n+2)}\right|\left|\frac{3^n(n+1)}{(x-2)^n}\right| = \frac{|x-2|}{3}\frac{(n+1)}{(n+2)},$$

whose limit is $|x-2|/3$ as $n \to \infty$. So the condition for guaranteed convergence is $|x-2|/3 < 1$, ie $|x-2| < 3$; thus the radius of convergence is 3 and the interval of convergence contains $(2-3, 2+3) = (-1, 5)$. Now we need to check the convergence at the endpoints of the interval.

When $x = -1$, the series is $\sum_n \frac{(-3)^n}{3^n(n+1)} = \sum_n \left(\frac{-3}{3}\right)^n \frac{1}{n+1} = \sum_n (-1)^n \frac{1}{n+1}$. This series converges by the alternating series test, since $1/(n+1)$ is decreasing and has limit 0 as $n \to \infty$.

When $x = 5$, the series is $\sum_n \frac{3^n}{3^n(n+1)} = \sum_n \frac{1}{n+1}$, which diverges for example by the p-test.

So the interval of convergence is $[-1, 5)$.

14.(10pts) It can be shown that

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n}, \text{ for } -1 < x < 1.$$

Note

- There is no need to check that the above formula is correct.
- Note that the above sum starts at 1.

(a) Use the above series to find an alternating series which sums to the following definite integral:

$$\int_0^{1/10} \ln(1+x^2)\,dx.$$

(If your answer to part (a) is given in expanded form without Σ-notation, please include the general form of the n-th term in your answer.)

We have

$$\begin{aligned}\int_0^{1/10} \ln(1+x^2)dx &= \int_0^{1/10} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{n} dx \\ &= \sum_{n=1}^{\infty} \int_0^{1/10} \frac{(-1)^{n+1}x^{2n}}{n} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^{1/10} x^{2n} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^{2n+1} n(2n+1)},\end{aligned}$$

where we used that $\int_0^{1/10} x^{2n}dx = \frac{1}{2n+1}x^{2n+1}|_0^{1/10} = \frac{1}{10^{2n+1}(2n+1)}$.

(b) Use the alternating series estimation theorem to estimate the value of $\displaystyle\int_0^{1/10} \ln(1+x^2)\,dx$ so that the error of estimation is less than $\dfrac{1}{10^5}$.

Let $b_n = 1/(10^{2n+1}n(2n+1))$, so that the desired quantity is $\sum_{n=1}^{\infty}(-1)^n b_n$. Note that $b_2 \leqslant 1/10^{2\cdot 2+1} = 1/10^5$. So taking the sum up to $n+1 = 2$, ie up to $n = 1$, will give the desired approximation. So we can take an estimation of $b_1 = -1/(3 \cdot 10^3) = -1/3000$.

True-False.

15.(6pts) Please circle "TRUE" if you think the statement is true, and circle "FALSE" if you think the statement is False.

(a)(1 pt. No Partial credit) The series $\sum_{n=1}^{\infty} \frac{10^n}{e^{n^2}}$ diverges by the n-th root test.

This is false, since the nth root is $10/e^n$ which tends to 0.

(b)(1 pt. No Partial credit) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.

This is false by p-series.

(c)(1 pt. No Partial credit) If $f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{2^{n+1} n!}$, then $f(2) = 1/2$. (Note: $0! = 1$)

This is true.

(d)(1 pt. No Partial credit) If $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n+2)}$, then the graph of $f(x)$ is increasing at $x = 0$.

We have $f'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2} x^{n-1}$, so that $f'(0) = -1/2 < 0$, so this is false.

(e)(1 pt. No Partial credit) The series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

This is true, for example by the divergence test.

(f)(1 pt. No Partial credit) If $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{nx^n}{(n+2)}$, then $f^{(5)}(0) = \frac{-5}{(5!)(7)}$

We have $f^{(5)}(0)/5! = (-1)^5 \cdot 5/7$, so $f^{(5)}(0) = -5! \cdot 5/7$, so this is false.

16.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: ~~(a)~~ (<u>not</u> an O around your answer choice) .

 Initials: ____________

The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Trapezoidal Rule:

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Simpson's Rule:

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Initials: ____________

ROUGH WORK